THEOREMS FOR CONDITIONAL EXPECTATIONS, WITH APPLICATIONS TO MARKOV PROCESSES

BY

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ABSTRACT

Several limit theorems analogous to Hopf's ergodic theorem (but where the usual ratio is replaced by conditional expectation with respect to certain sigma-fields) are proved and applications to probability theory are given.

1. The problem

Classical ergodic theorems concern averages of iterates of functions under operators derived from measure preserving transformations. This article is devoted to a study of averages of conditional expectations of such iterates. Besides having independent interest, these results are applicable to ratio-limit theorems in probability theory.

Let (Ω, Σ, π) be a σ-finite measure space, T a one-to-one invertible measure-preserving point transformation, and $E(\cdot \mid \Delta)$ the conditional expectation operator on $L_1(\Omega, \Sigma, \pi)$ where Δ is a sub σ-field of Σ, σ-finite with respect to π. To see the problem in its simplest setting, let T be conservative and ergodic and $\pi(\Omega) = 1$. Let T also denote the operator on L_1 defined by $(Tf)(\omega) = f(T\omega)$. For positive $f \in L_1$ the ergodic theorem shows

$$(1.1) \qquad Ef = E\left(\lim_{n\to\infty} n^{-1} \cdot \sum_{k=1}^{n} T^k f \,\middle|\, \Delta\right) \leqq \liminf_{n\to\infty} E\left(n^{-1} \cdot \sum_{k=1}^{n} T^k f \,\middle|\, \Delta\right)$$

and

$$(1.2) \qquad E \liminf_{n\to\infty} E\left(n^{-1} \cdot \sum_{k=1}^{n} T^k f \,\middle|\, \Delta\right) \leqq \liminf_{n\to\infty} E\left\{E\left(n^{-1} \cdot \sum_{k=1}^{n} T^k f \,\middle|\, \Delta\right)\right\} = Ef.$$

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Take the expectation of both sides in (1.1) and combine with (1.2) to obtain

$$Ef = \liminf_n E\left(n^{-1} \cdot \sum_{k=1}^{n} T^k f \,|\, \Delta\right).$$

However, Professor D. L. Burkholder has pointed out that the construction in [1, p. 888] produces an f and a Δ where

$$\limsup_n E\left(n^{-1} \cdot \sum_{k=1}^{n} T^k f \,|\, \Delta\right) = +\infty. \tag{1.3}$$

f must be unbounded for (1.3) to hold, otherwise the dominated convergence theorem and the ergodic theorem give a trivial proof of the convergence of the conditional expectation of the iterates to the constant Ef.

The general question, then, for a conservative, ergodic transformation T, a σ-finite measure π, and σ-fields Δ and Λ both σ-finite with respect to π, is: find conditions under which

$$E\left(\sum_{k=1}^{n} T^k f \,|\, \Delta\right) \Big/ E\left(\sum_{k=1}^{n} T^k g \,|\, \Lambda\right) = h_n(f, g, \Delta, \Lambda) \to \frac{Ef}{Eg} \text{ a.e.} (\pi) \tag{1.4}$$

for $f \in L_1$ and $g \geqq 0$ in L_1, $Eg > 0$. The example above shows that, although by Hopf's theorem

$$E\left(\sum_{k=1}^{n} T^k f\right) \Big/ \left(\sum_{k=1}^{n} T^k g\right) \to \frac{Ef}{Eg} \text{ a.e.}(\pi)$$

some restriction is already necessary even in the case of finite π.

In Section 3 a few theorems are proved whose conclusion states that (1.4) holds. But the most useful and interesting result is Theorem 3.5 in which the convergence obtained is not a.e. but only in measure. It is this theorem which has a particularly valuable application to Markov processes given in Section 4.

2. Notation and assumptions

We consider the background described at the beginning of the second paragraph of the preceding section. Restrictions which are to hold throughout Section 3 will now be imposed and certain notation defined.

Our basic restriction is on the type of σ-field Δ considered.

DEFINITION. A σ-finite σ-field Δ (with respect to π) is *admissible* if $T\Delta \subset \Delta$. ($T\Delta$ is the σ-field defined by: $E \in T\Delta$ if and only if $T^{-1}E \in \Delta$).

For any integer k, $T^k\Delta$ is a σ-field defined in the obvious way and is a sub-

σ-field of Σ. σ-finiteness of Δ and the one-to-one measure preserving property of T imply σ-finiteness of $T^k\Delta$. If Δ is admissible, $\{T^k\Delta, k \geqq 0\}$ is a decreasing sequence of σ-finite σ-fields with $\bigcap_{k \geqq 0} T^k\Delta = \Delta^*$, but Δ^* need not itself be σ-finite (and in the most interesting cases will not be).

Take $f \in L_1(\pi)$, Δ admissible, and $k \geqq 0$. Set $T^k\Delta = \Delta_k$ $(\Delta_0 = \Delta)$ and $f_k = E(f \mid \Delta_k)$. f_k is a backward martingale with respect to Δ_k (see [10]). If Δ^* is σ-finite, $\lim_{k\to\infty} f_k = E(f \mid \Delta^*)$ a.e., if Δ^* has only sets of trivial measure, $\lim_{k\to\infty} f_k = 0$ a.e. ([4], [14]). Δ-measurability of f implies $T^{-k}\Delta$ measurability of $T^k f$; an easy computation using the invariance of T under π shows

$$T^k E(f \mid \Delta) = E(T^k f \mid T^{-k}\Delta) \tag{2.1}$$

for any integer k. Thus we may write

$$E\left(\sum_{k=1}^{n} T^k f \mid \Delta\right) = \sum_{k=1}^{n} T^k f_k.$$

Our assumptions which hold throughout the paper are as follows.

(I) Only admissible σ-fields are considered.

(II) T is conservative, that is, the dissipative part is π-null [5].

(III) T is ergodic, that is, the σ-field of invariant sets ($T^{-1}E = E$ up to π equivalence) is trivial.

(IV) $\pi(\Omega) = \infty$.

(V) In ratios of the form (1.4), $f \in L_1$ and $g \geqq 0$ is in L_1 with $Eg > 0$.

Some of these restrictions are made simply to focus on the interesting questions and to avoid long-winded comments about trivial cases.

3. Main results

Recall the restrictions (I)–(V) imposed in the preceding section.

THEOREM 3.1. *Let* $\Lambda = \Delta$ *in* (1.4). *Then* (1.4) *is true.*

PROOF. Define a transformation S by

$$Sf = E(Tf \mid \Delta).$$

S is an L_1 operator of norm 1. Admissibility, (2.1) and properties of conditional expectation show

$$S^k f = T^k E(f \mid \Delta_k) = T^k f_k.$$

Then

$$h_n(f,g,\Delta,\Delta) = \left(\sum_{k=1}^{n} S^k f\right) \Big/ \left(\sum_{k=1}^{n} S^k g\right)$$

converges by the Chacon-Ornstein theorem [2] proving the convergence of the ratio in (1.4). The limit can be identified from [3].

THEOREM 3.2. *Let* $\bigcap_{k\geqq 0} T^k\Delta = \Delta^*$ *and* $\bigcap_{k\geqq 0} T^k\Lambda = \Lambda^*$ *both be σ-finite. Then* (1.4) *is true.*

PROOF. $T\Delta^* \subset T\Delta_n = \Delta_{n+1}$ for each n, and then $T\Delta^* \subset \Delta^*$. Similarly, $T^{-1}\Delta^* \subset \Delta^*$. Therefore $T\Delta^* = \Delta^*$ and also $T\Lambda^* = \Lambda^*$. Let u and v be non negative functions in L_1 measurable with respect to Δ^* and Λ^* respectively. It follows from the preceding that $T^k u$ and $T^k v$ are measurable with respect to Δ^* and Λ^* respectively for each integer k. Then

$$h_n(f,g,\Delta,\Lambda) = h_n(f,u,\Delta,\Delta)\cdot h_n(u,v,\Delta,\Lambda)\cdot h_n(v,g,\Lambda,\Lambda).$$

The first and third terms converge to Ef/Eu and Ev/Eg respectively by Theorem 3.1, and the middle term is simply

$$\left(\sum_{k=1}^{n} T^k u\right) \Big/ \left(\sum_{k=1}^{n} T^k v\right)$$

which converges to Eu/Ev by Hopf's theorem. The proof is complete.

Recall that $\Delta_k = T^k\Delta$ for any integer k and that $T^k u_{k-1} = E(T^k u \mid \Delta_{-1})$ for $u \in L_1$ (see 2.1).

LEMMA 3.3. *If*

$$\lim_{n\to\infty} \left(\left(\sum_{k=1}^{n} T^k u_{k-1}\right) \Big/ \left(\sum_{k=1}^{n} T^k u_k\right)\right) = 1 \quad \text{a.e. } (\pi) \tag{3.1}$$

for some $u \geqq 0$ *in* L_1, *then* (1.4) *holds for* $\Delta = \Delta_r$ *and* $\Lambda = \Delta_s$, r *and* s *fixed integers.*

PROOF. In each of the sums below, the summation is always on the index k from 1 to n, and r and s are fixed integers. We have

$$\frac{\sum T^k f_{k-1}}{\sum T^k g_k} = \frac{\sum T^k f_{k-1}}{\sum T^k u_{k-1}} \cdot \frac{\sum T^k u_{k-1}}{\sum T^k u_k} \cdot \frac{\sum T^k u_k}{\sum T^k g_k} .$$

The first and third terms converge by Theorem 3.1, and the middle term tends to 1, proving the lemma if $r = -1$, $s = 0$.

If $s = r + 1$, we have

$$h_n(f,g,\Delta_r,\Delta_s) = \frac{\sum T^k f_{k+r}}{\sum T^k g_{k+s}}.$$

Then

$$(3.2) \qquad T^{r+1}h_n(f,g,\Delta_r,\Delta_s) = \frac{\sum T^k(T^{r+1}f)_{k-1}}{\sum T^k(T^{r+1}g)_{k+s-r-1}}$$

$$= \frac{\sum T^k(T^{r+1}f)_{k-1}}{\sum T^k(T^{r+1}g)_k} \to \frac{ET^{r+1}f}{ET^{r+1}g}$$

$$= \frac{Ef}{Eg}$$

by the preceding, and then $h_n(f,g,\Delta_r,\Delta_s)$ itself converges by applying $T^{-(r+1)}$ to each side of (3.2). In the general case, when $s = r + a$, $a > 0$, say,

$$\frac{\sum T^k f_{k+r}}{\sum T^k g_{k+s}} = \frac{\sum T^k f_{k+r}}{\sum T^k f_{k+r+1}} \cdot \frac{\sum T^k f_{k+r+1}}{\sum T^k f_{k+r+2}} \cdot \frac{\sum T^k f_{k+r+a-1}}{\sum T^k g_{k+s}}$$

each term converging by the preceding, and the desired conclusion follows immediately.

THEOREM 3.4. *Each one of the following conditions is sufficient for* (1.4) *to hold for r and s fixed integers,* $\Delta = \Delta_r$, $\Lambda = \Delta_s$: *there exists* $u \geqq 0$ *in* $L_1 \cap L_2$ *such that*

$$(3.3) \qquad E \sup_{k \geqq 0} |u_k - u_{k-1}| < \infty,$$

$$(3.4) \qquad \liminf_{n\to\infty} n^{-\frac{1}{2}}\left(\sum_{k=1}^{n} T^k u\right) > 0 \qquad \text{a.e.}(\pi),$$

$$(3.5) \qquad \liminf_{n\to\infty} n^{-\frac{1}{2}}\left(\sum_{k=1}^{n} T^k u_k\right) > 0 \qquad \text{a.e.}(\pi).$$

PROOF. $\{u_n\}$ is a backward martingale with respect to $\{\Delta_n, n \geqq 0\}$ and the martingale differences $\{u_n - u_{n-1}\}$ are orthogonal. Thus

$$u_0 = u_n + (u_{n-1} - u_n) + (u_{n-2} - u_{n-1}) + \cdots + (u_0 - u_1)$$

so that, for all n

$$(3.6) \quad \sum_{k=1}^{n} E|u_k - u_{k-1}|^2 = E|u_0 - u_n|^2 = Eu_0^2 - Eu_n^2 \leqq Eu_0^2 \leqq Eu^2 < \infty$$

by properties of conditional expectation and the fact that $\{u_0, u\}$ is a submartingale with respect to $\{\Delta, \Sigma\}$. Hence the series $\Sigma_{k=1}^{\infty} E|u_k - u_{k-1}|^2$ converges, and by the measure-preserving property so does the series $\Sigma_{k=1}^{\infty} E\,|\,T^k(u_k - u_{k-1})^2\,|$. Then $\Sigma_{k=1}^{\infty} T^k(u_k - u_{k-1})^2$ converges a.e. (π). By the Cauchy-Schwartz inequality

$$\sum_{k=1}^{n} T^k \,|\, u_k - u_{k-1} \,| \leqq n^{\frac{1}{2}} \cdot \left\{ \sum_{k=1}^{n} T^k \,|\, u_k - u_{k-1} \,|^2 \right\}^{\frac{1}{2}}$$

and by the above, this implies

$$n^{-\frac{1}{2}} \cdot \sum_{k=1}^{n} T^k \,|\, u_k - u_{k-1} \,| \to 0. \tag{3.7}$$

Divide (3.7) by $n^{-\frac{1}{2}} \cdot \Sigma_{k=1}^{n} T^k u_k$. Under (3.5) the result of this division tends to zero and Lemma 3.3 can be applied to obtain the desired conclusion. (3.4) easily implies (3.5) by taking conditional expectation and using Fatou's theorem, so again the conclusion follows. To prove sufficiency of (3.3), observe that Hopf's theorem yields for all $s \geqq 1$

$$\lim_{n \to \infty} \left(\left(\sum_{k=s}^{n} T^k \sup_{j \geqq s} \,|\, u_j - u_{j-1} \,| \right) \Big/ \left(\sum_{k=1}^{n} T^k u \right) \right) = \left(E \sup_{j \geqq s} \,|\, u_j - u_{j-1} \,| \right) \Big/ Eu.$$

Without loss of generality, Δ^* may be assumed to contain sets only of measure zero or ∞, for one may extract a largest invariant, σ-finite part of Δ^* on which Theorem 3.2 applies. Then $\lim_{n \to \infty} u_n = 0$ a.e. and by (3.3)

$$\lim_{s \to \infty} E \sup_{j \geqq s} \,|\, u_j - u_{j-1} \,| = 0.$$

The divergence of the denominator yields

$$\limsup_{n \to \infty} \left(\left(\sum_{k=1}^{n} T^k |u_k - u_{k-1}| \right) \Big/ \sum_{k=1}^{n} T^k u \right)$$

$$\leqq \lim_{s \to \infty} \lim_{n \to \infty} \left(\left(\sum_{k=s}^{n} T^k \sup_{j \geqq s} \,|\, u_j - u_{j-1} \,| \right) \Big/ \left(\sum_{k=1}^{n} T^k u \right) \right) = 0.$$

It follows that

$$\lim_{n \to \infty} \left(\left(1 + \left[\sum_{k=1}^{n} T^k(u_k - u_{k-1}) \right]^{+} \right) \Big/ \left(\sum_{k=1}^{n} T^k u \right) \right)$$

$$= \lim_{n \to \infty} \left(\left(1 + \left[\sum_{k=1}^{n} T^k(u_k - u_{k-1}) \right]^{-} \right) \Big/ \left(\sum_{k=1}^{n} T^k u \right) \right) = 0. \tag{3.8}$$

For each N,

$$E\left(\inf_{n \geq N} \left(\left(\sum_{k=1}^{n} T^k u \right) \Big/ \left(1 + \left[\sum_{k=1}^{n} T^k (u_k - u_{k-1}) \right]^+ \right) \right) \Big| \Delta_{-1} \right)$$

$$\leq \inf_{n \geq N} \left(\left(\sum_{k=1}^{n} T^k u_{k-1} \right) \Big/ \left(1 + \left[\sum_{k=1}^{n} T^k (u_k - u_{k-1}) \right]^+ \right) \right) \tag{3.9}$$

and the monotone convergence theorem with (3.8) shows that the right side of (3.9) converges to ∞ as $N \to \infty$. A similar argument about the negative part proves (3.1), and so Lemma 3.3 applies. The proof is complete.

THEOREM 3.5. *Let $f \in L_1$. Suppose there exists an increasing sequence of positive constants h_n tending to ∞ with*

$$\limsup_{n \to \infty} \left(\left(\sum_{k=1}^{n} T^k f_k \right) \Big/ h_n \right) < \infty \qquad \text{a.e. } (\pi) \tag{3.10}$$

and

$$\lim_{n \to \infty} \left(\left(\sum_{k=1}^{n} T^k f_k \right) \Big/ h_n^{\frac{1}{2}} \right) = \infty \qquad \text{a.e. } (\pi). \tag{3.11}$$

Then if α is a finite measure equivalent to π, and r and s are fixed

$$\lim_{n \to \infty} \left(\left(E \left(\sum_{k=1}^{n} T^k f \,|\, \Delta_r \right) \right) \Big/ \left(E \left(\sum_{k=1}^{n} T^k g \,|\, \Delta_s \right) \right) \right) = \frac{Ef}{Eg} \text{ in } \alpha\text{-measure.} \tag{3.12}$$

Consequently, in the notation of (1.4), *given any sequence $\{n_i\}$, there is a further subsequence $\{m_j\}$ with*

$$h_{m_j}(f, g, \Delta_r, \Delta_s) \to \frac{Ef}{Eg} \qquad \text{a.e. } (\pi).$$

If (3.10) *and* (3.11) *are replaced by*

$$\lim_{n \to \infty} \left(\left(\sum_{k=1}^{n} T^k f_k \right) \Big/ h_n \right) = 0 \quad \text{a.e. } (\pi) \tag{3.10'}$$

and

$$\liminf_{n \to \infty} \left(\left(\sum_{k=1}^{n} T^k f_k \right) \Big/ h_n^{\frac{1}{2}} \right) > 0 \quad \text{a.e. } (\pi) \tag{3.11'}$$

respectively, then (3.12) *is still valid.*

PROOF. Notice that by Theorem 3.1, if (3.10) and (3.11) are true for any $f \in L_1$, they are true for all $f \in L_1$. Let $f = 1_A$ be the indicator of a Δ-measurable set of finite measure. Recalling that $T^k f_k$ is Δ-measurable, and $T^k f_{k-1}$ is Δ_{-1} measurable for each k, by admissibility and properties of conditional expectation we have, for $1 \leq j < k$

$$ET^j f_j T^k f_{k-1} = E(E(T^j f_j T^k f_{k-1} \mid \Delta)) = ET^j f_j E(T^k f_{k-1} \mid \Delta) = ET^j f_j T^k f_k.$$

Using the invariance of the integral this implies

$$(3.13)\quad ET^j(f_j - f_{j-1})T^k(f_k - f_{k-1}) = ET^{j-1}f_{j-1}T^{k-1}f_{k-1} - ET^j f_j T^k f_k.$$

Now we compute

$$(3.14)\quad \begin{aligned} E\Big(\sum_{k=1}^{n} T^k(f_k - f_{k-1})\Big)^2 &= E\Big(\sum_{k=1}^{n} T^k(f_k - f_{k-1})^2\Big) \\ &\quad + 2 \sum_{1 \leqq j < k \leqq n} ET^j(f_j - f_{j-1})T^k(f_k - f_{k-1}). \end{aligned}$$

The first term on the right of (3.14) converges by (3.6) and the second term is easily seen to be

$$2\left[E(T^0 f_0)\Big(\sum_{j=1}^{n-1} T^j f_j\Big) - E(T^n f_n)\Big(\sum_{j=1}^{n-1} T^j f_j\Big)\right],$$

by (3.13). Thus

$$(3.15)\quad E\Big(\sum_{k=1}^{n} T^k(f_k - f_{k-1})\Big)^2 \leqq c + 2E(T^0 f_0)\Big(\sum_{j=1}^{n} T^j f_j\Big) = c + 2\int_A \sum_{j=1}^{n} T^j f_j,$$

where c is constant. Without loss of generality, we may assume

$$(3.16)\quad 2\Big(\sum_{k=1}^{n} T^k f_k\Big)\Big/ h_n < L < \infty$$

everywhere on A for $n \geqq N$, N taken large enough. This follows by (3.10); for then (3.10) holds on a subset $B \subset A$ of positive π measure for $n \geqq N$. Replacing all of the above arguments with $g = 1_B$, the last term on the right in (3.15) gives

$$2\int_B \sum_{j=1}^{n} T^j g_j$$

and (3.16) holds everywhere on B. Hence we may and do assume (3.16) is true on A.

For $\varepsilon > 0$ fixed, define

$$U_n = \left\{\Big(\sum_{k=1}^{n} T^k(f_k - f_{k-1})\Big)^2 \geqq \varepsilon\Big(\sum_{k=1}^{n} T^k f_k\Big)^2\right\}.$$

By (3.15) and (3.16)

$$(3.17)\quad \varepsilon\int_{U_n}\left(\Big(\sum_{k=1}^{n} T^k f_k\Big)^2\Big/ h_n\right) \leqq E\Big(\sum_{k=1}^{n} T^k(f_k - f_{k-1})\Big)^2\Big/ h_n < (L+1)\pi(A)$$

for large n. Moreover, for fixed $\lambda > 0$, set

$$S_m = \left\{\left(\sum_{k=1}^{n} T^k f_k\right)^2 \Big/ h_n > ((L+1)\pi(A))\lambda^{-1} \text{ for } n \geqq m\right\}.$$

Then (3.17) implies $\pi(S_m \cap U_n) < \lambda/\varepsilon$ for $n \geqq m$. Let α be equivalent to π and finite. By (3.11), m can be taken so large that on S'_m, the complement of S_m, $\alpha(S'_m) < \delta$ whatever λ is; in addition λ may be taken so small that by absolute continuity $\alpha(S_m \cap U_n) < \delta$. Thus $\alpha(U_n) < 2\delta$ for all large n. Since δ is arbitrary $\alpha(U_n) \to 0$, that is

$$\left(\sum_{k=1}^{n} T^k f_{k-1}\right) \Big/ \left(\sum_{k=1}^{n} T^k f_k\right) \to 1 \text{ in } \alpha\text{-measure.}$$

A simple adaptation of Lemma 3.3 yields the desired conclusion. If (3.10′) and (3.11′) hold, (3.17) reduces to (for large n)

$$a\varepsilon\pi(U \cap U_n) \leqq \varepsilon \int_{U \cap U_n} \left(\left(\sum_{k=1}^{n} T^k f_k\right)^2 \Big/ h_n\right) \leqq E\left(\sum_{k=1}^{n} T^k(f_k - f_{k-1})\right)^2 \Big/ h_n \to 0$$

for some $\alpha > 0$ and for arbitrarily large U with $\pi(U) < \infty$. Again, this implies $\alpha(U_n) \to 0$ and the proof is concluded as before.

REMARK. If $h_n = n$, (3.10′) automatically holds by the ergodic theorem when $\pi(\Omega) = \infty$. If (3.11′) holds for $h_n = n$, then we are in the situation (3.5) of Theorem 3.4 and obtain convergence a.e. Thus Theorem 3.5 can be viewed as a generalization of Theorem 3.4.

4. Application to probability theory

Markov processes

Let $\{X_n, -\infty < n < +\infty\}$ be a Markov process on a state space S with stationary transition probabilities $P^n(x, E) = P(X_n \in E \mid X_0 = x)$ and σ-finite stationary measure λ. We assume the process is recurrent in the following sense; the process satisfies condition B which states that $\lambda(E) > 0$ implies $P(X_n \in E$ infinitely often $\mid X_0 = x) = 1$ a.e. (λ) on S.

A stronger, related condition is that of Harris [6]: Condition C is the same as Condition B except that the phrase "a.e. (λ) on S" is replaced by "for all $x \in S$". Condition C will not be assumed unless explicitly stated.

λ induces a measure π in a natural way on bilateral coordinate space (with product σ-field); that λ is stationary entails the invariance of π with respect to the shift transformation T (see, for example, [7]). This is the key correspondence relating Section 3 to the theory of Markov processes. Condition B is equivalent to the statement that T and π on coordinate space satisfy (II) and (III).

Let $\Delta = \mathscr{B}(\cdots X_{-1}, X_0)$ where $\mathscr{B}(\cdot)$ is the σ-field generated by the set of random variables in parenthesis. Then $T\Delta = \mathscr{B}(\cdots, X_{-2}, X_{-1}) \subset \Delta$ and Δ being σ-finite, is admissible. $\Delta^* = \tau_{-\infty}$ is the left tail σ-field of the process.

Let $f = 1_{X_0 \in E}$, the X_0-indicator of a set E, $0 < \lambda(E) < \infty$. Then

$$T^k f_k = E(T^k f \mid \Delta) = E(T^k f \mid \cdots X_{-1}, X_0) = E(1_{X_k \in E} \mid X_0) = P^k(X_0, E) \tag{4.1}$$

by the Markov property (see also [10]). We then have for $r > 0$

$$\left(\sum_{k=1}^{n} E(T^k f \mid \Delta_0)\right) \Big/ \left(\sum_{k=1}^{n} E(T^k f \mid \Delta_{-r})\right) \sim \left(\sum_{k=1}^{n} P^k(X_0, E)\right) \Big/ \left(\sum_{k=1}^{n} P^k(X_r, E)\right).$$

From [11] it follows that

$$\liminf_{n \to \infty} \left(\left(\sum_{k=1}^{n} P^k(X_0(\omega), E)\right) \Big/ \left(\sum_{k=1}^{n} P^k(t, E)\right) \right) = \text{constant a.e. } (\pi)$$

for each fixed t, E, when Condition B holds. A similar result can be proved for lim sup. From Theorem 3.5 we obtain Theorem 4.1.

THEOREM 4.1. *If for some* $t \in S$, *we have* a.e. (π)

$$\begin{aligned} 0 &< \liminf_{n \to \infty} \left(\left(\sum_{k=1}^{n} P^k(X_0, E)\right) \Big/ \left(\sum_{k=1}^{n} P^k(t, E)\right) \right) \\ &\leqq \limsup_{n \to \infty} \left(\left(\sum_{k=1}^{n} P^k(X_0, E)\right) \Big/ \left(\sum_{k=1}^{n} P^k(t, E)\right) \right) < \infty \end{aligned} \tag{4.2}$$

then

$$\lim_{n \to \infty} \left(\left(\sum_{k=1}^{n} P^k(X_0, E)\right) \Big/ \left(\sum_{k=1}^{n} P^k(X_r, F)\right) \right) = \frac{\lambda(E)}{\lambda(F)} \text{ in } (\alpha) \text{ measure,} \tag{4.3}$$

for each fixed integer r, *where* α *is a finite measure equivalent to* π, *and* E *and* F *are sets with* $0 < \lambda(E), \lambda(F) < \infty$. *Consequently, there is a subsequence* $\{m_j\}$ *for each given sequence* $\{n_i\}$ *with*

$$\left(\left(\sum_{k=1}^{m_j} P^k(X_0, E)\right) \Big/ \left(\sum_{k=1}^{m_j} P^k(X_r, F)\right) \right) = \frac{\lambda(E)}{\lambda(F)} \quad \text{a.e. } (\pi). \tag{4.4}$$

PROOF. Putting $h_n = \sum_{k=1}^{n} P^k(t, E)$ in Theorem 3.5 we have (3.10) and (3.11) holding. Then (4.3) is simply (3.12) in a particular case.

For $x \in S$, put $\nu_x(\cdot) = \sum_{k=1}^{\infty} \frac{1}{2^k} P^k(x, \cdot)$.

COROLLARY 4.2. *If* (4.2) *holds, for each sequence* $\{n_i\}$ *there is a further subsequence* $\{m_j\}$ *with*

$$\lim_{j\to\infty} \left(\left(\sum_{k=1}^{m_j} P^k(x,E)\right) \Big/ \left(\sum_{k=1}^{m_j} P^k(y,F)\right)\right) = \frac{\lambda(E)}{\lambda(F)} \tag{4.5}$$

for almost all $y(\nu_x)$ *for all* x *in a* λ*-full set.*

PROOF. As ω ranges through a π-full set, $X_0(\omega)$ ranges through a λ-full set of x and then $X_r(\omega)$ for fixed $r \geqq 1$ ranges through a $P^r(x, \cdot)$-full set of y. Hence for almost all $x(\lambda)$, by taking a diagonal subsequence, (4.3) holds for all $r \geqq 1$, and so (4.5) holds for almost all $y(\nu_x)$.

THEOREM 4.3. *Let* $\sum_{k=1}^{n} P^k(t,E) \to \infty$ *where* $\lambda(E) > 0$. *Suppose, for some* x,

$$\liminf_{n\to\infty} \left(\left(\sum_{k=1}^{n} P^k(x,E)\right) \Big/ \left(\sum_{k=1}^{n} P^k(t,E)\right)\right) = 0, \tag{4.6}$$

then there is a subsequence $\{m_j\}$ *with*

$$\lim_{j\to\infty} \left(\left(\sum_{k=1}^{m_j} P^k(y,E)\right) \Big/ \left(\sum_{k=1}^{m_j} P^k(t,E)\right)\right) = 0 \quad \text{a.e. } (\nu_x), \tag{4.7}$$

and there is a decreasing sequence of sets $E_i \subset E$ *with* $\nu_x(\cap_i E_i) = 0$ *and*

$$\limsup_{n\to\infty} \left(\left(\sum_{k=1}^{n} P^k(t,E_i)\right) \Big/ \left(\sum_{k=1}^{n} P^k(t,E)\right)\right) = 1, \tag{4.8}$$

for all i.

PROOF. By hypothesis there is a subsequence $\{n_i\}$ such that, for each fixed r,

$$0 \leftarrow \left(\sum_{k=1}^{n_i+r} P^k(x,E)\right) \Big/ \left(\sum_{k=1}^{n_i} P^k(t,E)\right)$$

$$= \int P^r(x,dy)\left(\sum_{k=1}^{n_i} P^k(y,E)\right) \Big/ \left(\sum_{k=1}^{n_i} P^k(t,E)\right)\Bigg).$$

The sequence in the integrand converges in L_1 to zero with respect to $P^r(x, \cdot)$-measure. Use the diagonal method to find a subsequence, say $\{m_j\}$, converging a.e. $P^r(x, \cdot)$ for all r, hence converging to zero a.e. (ν_x). Now define

$$F_A^j(x,E) = P(X_j \in E;\ X_i \notin A,\ 0 < i < j \mid X_0 = x), \quad k \geqq 2$$

$$= P(x,E), \qquad k = 1$$

and

$$\sum_{j=1}^{\infty} F_A^j(x, E) = P_A(x, E).$$

If $E \subset A$, $P_A(x, E) \leqq 1$ and is the expected number of visits to E in one visit to A, starting from x (see [12, p. 645]). Let $\varepsilon > 0$ be given and let A_l be an increasing sequence of sets converging to a v_x-full set on which the ratio in (4.7) becomes and stays lesss than ε for large enough m_j. Then, by [12, p. 646], for any fixed A_l

$$\text{(4.9)} \qquad \sum_{l=1}^{m_j} P^l(t, E) = \int_{A_l} \sum_{k=1}^{m_j-1} \left\{ F_{A_l}^k(t, dy) \sum_{l=1}^{m_j-k} P^l(y, E) \right\} + \sum_{l=1}^{m_j} F_{A_l}^l(t, E).$$

Dividing and going to the limit shows

$$1 \leqq \varepsilon + \limsup_{j \to \infty} \left(\left(\sum_{l=1}^{m_j} F_{A_l}^l(t, E) \right) \Big/ \left(\sum_{l=1}^{m_j} P^l(t, E) \right) \right)$$

$$= \varepsilon + \limsup_{j \to \infty} \left(\left(\sum_{l=1}^{m_j} F_{A_l}^l(t, E - A_l) \right) \Big/ \left(\sum_{l=1}^{m_j} P^l(t, E) \right) \right)$$

$$\leqq \varepsilon + \limsup_{j \to \infty} \left(\left(\sum_{l=1}^{m_j} P^l(t, E - A_l) \right) \Big/ \left(\sum_{l=1}^{m_j} P(t, E) \right) \right).$$

Since ε is arbitrary, this proves the theorem when we put $E_l = E - A_l$.

Theorem 4.1, its Corollary 4.2, and Theorem 4.3 can be used in conjunction to prove (4.3) and (4.5) under very general conditions. The key relation one must show is (4.2). It is not difficult to see that all random walks on the line satisfying Condition B also satisfy (4.2). If Condition C is known to hold, it is possible to use our results to prove

$$\text{(4.10)} \qquad \left(\sum_{k=1}^{n} P^k(x, E) \right) \Big/ \left(\sum_{k=1}^{n} P^k(y, F) \right) \to \frac{\lambda(E)}{\lambda(F)}$$

for all x and y in a λ-full set, when $0 < \lambda(E), \lambda(F) < \infty$. This is a result of Jain [13]. Proving this in detail now would be tedious, but we sketch a proof of this theorem: under Condition C v_x-null sets are λ-null for all x. Thus if (4.6) holds, there is a sequence $\{E_l\}$ decreasing to a λ-null set, $E_l \subset E$, with (4.8) holding. According to [9, Th. 1], this can only happen for t on a *fixed* λ-null set (when E is fixed). (The proof in [9] depends, however, on the ratio-limit theorem (4.10), but an independent proof may be given of this fact). From this it will follow readily that (4.2) is true for almost all $t(\lambda)$. Thus the assertions of Theorem 4.1 and its Corollary 4.2 hold; this gives convergence of subsequences in (4.5) for all x and y

in a λ-full set. Once convergence on subsequences is known, it is not difficult to show convergence over the entire sequence.

Strictly stationary processes

This example is more general than the preceding. We consider processes whose finite dimensional measures (that is, distributions) are invariant under time shifts. The measures are permitted to be σ-finite. Such processes provide the most general probabilistic examples for which the results of Section 3 hold when T = the shift transformation.

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